Phys 410
Fall 2015
Lecture #27 Summary
3 December, 2015

We continued our discussion of Special Relativity. Einstein made two postulates:

1) If S is an inertial reference frame and if a second frame S' moves with constant velocity relative to S, then S' is also an inertial reference frame.
2) The speed of light (in vacuum) has the same value c in every direction in all inertial reference frames.

We considered the measurement of length (ℓ) in two different inertial reference frames. This led to the length contraction result: $\ell = \ell_0/\gamma$, where ℓ_0 is the 'proper length' of an object, namely the length when the object is at rest in your reference frame. All observers in other states of motion measure a contracted length.

We discussed the inadequacy of the Galilean transformation of coordinates between two different inertial reference frames S and S'. For example the translation of x-coordinates between two reference frames moving at speed V in the x-direction is $x' = x - Vt$. But the two observers cannot even agree on time, so this equation is of little use. We derived the relativistic version of this transformation between S and S' moving at speed V in the x-direction, making use of length contraction to arrive at the Lorentz transformation:

$$x' = \gamma(x - Vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t - xV/c^2)$$

Note that these equations reduce to the Galilean version in the limit $\frac{V}{c} \ll 1$. These equations show how a single event (in space-time) is described in two different inertial reference frames that are moving at a constant speed V relative to each other in the x-direction.

We applied the Lorentz transformation to the apparent paradox of a 100-cm-long relativistic snake moving across a table (at $\frac{V}{c} = 0.6$) that has two knives bouncing on the table simultaneously at a distance 100 cm apart. From the perspective of reference frame S at rest with respect to the table, the snake will be Lorentz contracted and easily fit between the falling knives. Naively, from the snake's perspective in frame S' the two knives appear to be only 80 cm apart, meaning that it will surely be cut in two by the two falling knives. This argument implicitly assumes that the knives also appear to bounce simultaneously from the snake's perspective. The resolution of this paradox is a careful evaluation of the locations in space and time of the two chopping knives in each reference frame using the Lorentz transformation. We found that in the snake's frame of reference the first knife just misses its tail, but the second knife falls 2.5 ns before the first and at a location of 125 cm in its frame, missing the snake altogether. Hence the paradox is resolved. However the results seem unsettling because the events that were simultaneous in S are no longer simultaneous in S'. In addition, the two knives appear to be too far apart in S'. These issues arise because we are used to dealing with situations where information travels much faster (at the speed of light!) compared to the motions of the objects of interest, and the distances covered in time Δt are very small compared to $c\Delta t$. Hence we can get a 'global' view of the system and ascribe a single universal time coordinate to the motion. This is no longer the case when objects are moving at speeds approaching light speed. It takes significant time for information to travel between two spatially separated points, and these delays must be incorporated into our description of the motion. To handle this, we will now develop a description of events in a four-dimensional space-time, and learn how to calculate the correct "invariant interval" between two events in space-time.

We deduced the relativistic velocity addition formula from the differential form of the (linear) Lorentz transformation. Velocities of objects measured in frames S' and S moving at relative speed V in the x-direction are related as $v_x' = \frac{v_x - V}{1 - V v_x/c^2}$, and $v_y' = \frac{v_y}{\gamma(1 - V v_x/c^2)}$, $v_z' = \frac{v_z}{\gamma(1 - V v_x/c^2)}$. For example if a spaceship is approaching earth at a speed of $\frac{V}{c} = 0.8$ and launches a light beam towards us ($v_x' = c$) then we measure the speed of that light as not $1.8\,c$, but as $v_x = c$, in accordance with the second postulate of relativity. The velocity addition formulas thus enforce the speed limit of the universe!

It was noted that the Lorentz transformation has the appearance of a rotation in a 4-dimensional space spanned by the coordinates x_1, x_2, x_3 (the re-named ordinary Cartesian coordinates) and a new coordinate $x_4 = ct$. The Lorentz transformation can be written in "rotational" form as $x'^{(4)} = \overline{\overline{\Lambda}}\, x^{(4)}$, where $x^{(4)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is the space-time 4-vector [which can also be written as $x^{(4)} = (\vec{x}, ct)$, for example] and the 'rotation' matrix representing the Lorentz transformation is $\overline{\overline{\Lambda}} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$. This is not the most general Lorentz transformation. It is a special case called a "boost", which corresponds to a pair of reference frames moving relative to each other along one of the coordinate axes (x_1). Note that we use the superscript $x^{(4)}$ to denote 4-vectors and the vector sign ($\vec{x}$) to denote ordinary 3-vectors.

One can define the rapidity as an angle obeying the equation $\tanh(\varphi) = \beta$, where β is the normalized relative velocity between the two reference frames, as always. With this definition, the Lorentz transformation matrix can be written as $\bar{\bar{\Lambda}} = \begin{pmatrix} \cosh(\varphi) & 0 & 0 & -\sinh(\varphi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\varphi) & 0 & 0 & \cosh(\varphi) \end{pmatrix}$, which bears a strong resemblance to a rotation matrix in 3-space, except for the use of hyperbolic functions (rather than trigonometric functions) and an extra minus sign. One can think of the Lorentz transformation as a rotation of the 4-space coordinate axes that are used to describe a specific physical event. One nice feature of the rapidity arises in velocity addition. Velocities do not simply add, as we know from the equations above, but must be combined in a rather peculiar way. On the other hand, rapidities do add linearly; if we add two velocities v_1 and v_2 along the x_1 axis to get the new velocity u, we must use the formula $u = \frac{\mathrm{v}_1+\mathrm{v}_2}{1+\mathrm{v}_1\mathrm{v}_2/c^2}$, whereas for rapidity one simply has $u = \tanh(\varphi_1 + \varphi_2)$. In other words, adding two relativistic velocities is like two consecutive rotations through angles φ_1 and φ_2.